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THE STABILITY OF THE GRIOLI PRECESSION†

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The motion of a rigid body about a fixed point in a uniform gravitational field is considered. The body is not dynamically symmetric, but its centre of gravity is on the perpendicular, erected from the fixed point, to a circular section of the inertia ellipsoid. Grioli proved that a rigid body with such mass geometry may precess regularly about a non-vertical axis. The problem of the stability of this precession is solved. © 2003 Elsevier Ltd. All rights reserved.

In 1947, Grioli [1] made an unexpected discovery: a rigid body without dynamic symmetry, moving in a uniform gravitational field, may precess regularly about an axis other than the vertical axis. Until then, regular precession of a heavy rigid body had been known in the Euler and Lagrange cases, when the body is dynamically symmetric, and statements were sometimes made in the literature [2, 3] to the effect that regular precession of a heavy rigid body about a non-vertical axis is impossible.

At the present time, the problem of the existence of regular precession of a rigid body with one fixed point in a uniform gravitational field has been fully solved. Studies have shown [4–7] that only three types of regular precession exist: (1) precession of a dynamically symmetric body in the Euler case about an arbitrary axis of fixed direction passing through the fixed point; (2) regular precession about the vertical in the Lagrange case; (3) regular precession, discovered by Grioli, of a body that is not dynamically symmetric, about an axis inclined to the vertical. In all types of regular precession, the centre of gravity of the body lies on a perpendicular to a circular section of its inertia ellipsoid for the fixed point. A history of the discovery and of the study of rigid body precession may be found in [8–10].

The problem of the stability of regular Grioli precession has proved to be very complex and, unlike precession in the classical Euler and Lagrange cases, still awaits a complete solution, though attempts have been made to investigate it [11–14].

Below we present new results on this topic, a brief summary was published in [15].

1. THE MOTION OF A RIGID BODY IN THE CASE OF GRIOLI PRECESSION

Consider the motion of a rigid body with one fixed point O in a uniform gravitational field. The weight of the body is mg and the distance from the centre of gravity to the fixed point is l. Suppose the point of attachment is chosen is such a way as to satisfy the following conditions (Grioli's conditions)

$$x'_g\sqrt{B-C} = z'_g\sqrt{A-B}, \quad y'_g = 0, \quad A>B>C \tag{1.1}$$

where x'_g, y'_g, z'_g, are the coordinates of the centre of gravity G in the system of coordinates $Ox'y'z'$ formed by the principal axes of inertia of the body for the fixed point, A, B and C being the corresponding moments of inertia. Conditions (1.1) mean that the body does not possess dynamic symmetry, and its centre of gravity lies on a perpendicular, erected from the fixed point, to a circular section of the inertia ellipsoid (Fig. 1).

Let $OXYZ$ be a fixed system of coordinates whose OZ axis points directly upward, while the system $Oxyz$ is rigidly attached to the body, its Oy axis coinciding with the principal axis of inertia Oy' of the body for the point O corresponding to the median moment of inertia B, and its Oz axis passing through the centre of gravity of the body. The trihedron $Oxyz$ is obtained by rotating the trihedron $Ox'y'z'$ by an angle α about the Oy' axis (Fig. 2a), where

$$\alpha = \operatorname{arctg}\frac{x'_g}{z'_g} = \operatorname{arctg}\sqrt{\frac{A-B}{B-C}}$$

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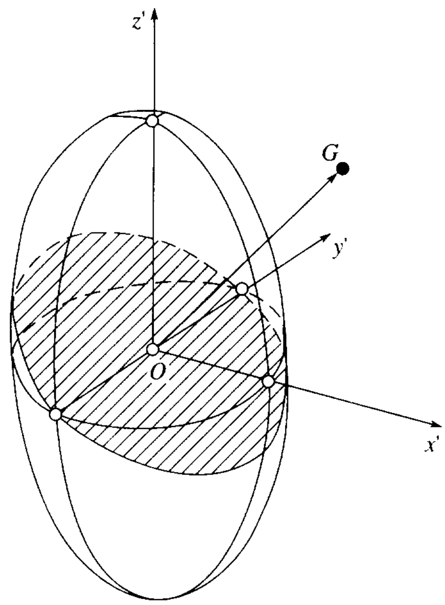

Fig. 1

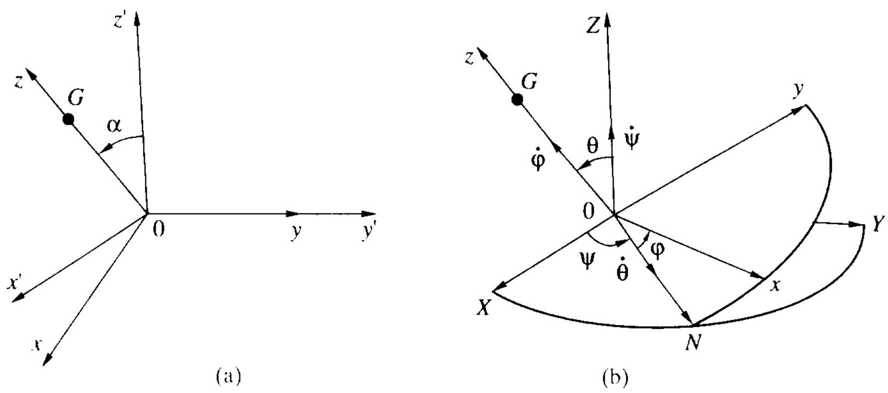

Fig. 2

In the system of coordinates $Oxyz$ the matrix $\mathbf{J}$ of the inertia tensor and the unit vector γ of the fixed vertical axis OZ are written as

$$\mathbf{J} = \begin{Vmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{Vmatrix}, \quad \boldsymbol{\gamma} = \begin{Vmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Vmatrix}$$

$$J_x = J_y = B, \quad J_z = A - B + C, \quad J_{xz} = -\sqrt{(A-B)(B-C)}$$

The kinetic and potential energies of the body are given by

$$T = 1/2 J_x (p^2 + q^2) + 1/2 J_z r^2 - J_{xz} pr, \quad \Pi = mgl\gamma_3 \tag{1.2}$$

where p, q and r are the projections of the angular velocity vector $\boldsymbol{\omega}$ of the body on the Ox, Oy and Oz axes, respectively.

As generalized coordinates q_1, q_2, q_3 defining the orientation of the trihedron $Oxyz$ relative to the fixed system of coordinates we take the Euler angles φ, θ, ψ, defined in the usual way (Fig. 2b). Then

$$p = \dot{q}_3\gamma_1 + \dot{q}_2\cos q_1, \quad q = \dot{q}_3\gamma_2 - \dot{q}_2\sin q_1, \quad r = \dot{q}_3\gamma_3 + \dot{q}_1$$
$$\gamma_1 = \sin q_2 \sin q_1, \quad \gamma_2 = \sin q_2 \cos q_1, \quad \gamma_3 = \cos q_2 \tag{1.3}$$

As generalized momenta we take the dimensionless quantities p_i ($i = 1, 2, 3$), defined by the equalities

$$p_i = \frac{1}{An}\frac{\partial T}{\partial \dot{q}_i}, \quad i = 1, 2, 3 \tag{1.4}$$

The kinetic energy T is evaluated by formulae (1.2) and (1.3), and n is defined by the relations

$$n^2 = \frac{mgl}{(A - B + C)\sqrt{b^2+1}}, \quad b = \frac{b_1}{b_2}, \quad b_1 = \sqrt{(1-\theta_b)(\theta_b - \theta_c)}, \quad b_2 = 1 - \theta_b + \theta_c$$

where $\theta_b = B/A$ and $\theta_c = C/A$ are the dimensionless parameters of the problem. The domain of their admissible values in the θ_b, θ_c plane ($0 < \theta_c < \theta_b < 1$, $\theta_b + \theta_c > 1$) is a right-angled triangle with vertices $P_1(^1/_2, {}^1/_2)$, $P_1(1, 1)$, $P_3(1, 0)$.

If we now additionally take as independent variable $\tau = n(t + t_0)$, where t_0 is an arbitrary constant, then, taking Eqs (1.2)–(1.4) into consideration, we obtain the following expression for the Hamiltonian

$$H = \frac{1}{2\theta_b\theta_c\sin^2 q_2}\sum_{k_1+k_2+k_3=2} a_{k_1k_2k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3} + \Pi^* \tag{1.5}$$

$$a_{200} = \theta_b(\theta_b \sin^2 q_2 + b_2\cos^2 q_2) - b_1^2\cos^2 q_1 \cos^2 q_2 + 2\theta_b b_1 \sin q_1 \sin q_2 \cos q_2$$

$$a_{110} = -2(b_1^2 \sin q_1 \cos q_1 \cos q_2 + \theta_b b_1 \cos q_1 \sin q_2)\sin q_2$$

$$a_{101} = -2(\theta_c \cos q_2 + b_1^2 \sin^2 q_1 \cos q_2 + \theta_b b_1 \sin q_1 \sin q_2)$$

$$a_{020} = (\theta_c + b_1^2\cos^2 q_1)\sin^2 q_2$$

$$a_{011} = 2b_1^2 \sin q_1 \cos q_1 \sin q_2$$

$$a_{002} = \theta_c + b_1^2 \sin^2 q_1$$

$$\Pi^* = \sqrt{b_1^2 + b_2^2}\cos q_2$$

The following solution of the equations of motion corresponds to Grioli precession

$$q_1 = f_1(\tau) = -\frac{\pi}{2} + \tau - \operatorname{arctg}(b\sin\tau), \quad p_1 = g_1(\tau) = b_2(1 - b\cos\tau) \tag{1.6}$$

$$q_2 = f_2(\tau) = \arccos\frac{b\cos\tau}{\sqrt{b^2+1}}, \quad p_2 = g_2(\tau) = \frac{b\sin\tau}{\sqrt{1+b^2\sin^2\tau}}(1 + \theta_c - bb_2\cos\tau) \tag{1.7}$$

$$q_3 = f_3(\tau), \quad p_3 = \frac{\theta_c}{b_2\sqrt{b^2+1}} \tag{1.8}$$

where

$$f_3(\tau) = q_3(0) + (2k+1)\frac{\pi}{2} - \operatorname{arctg}\frac{\operatorname{ctg}\tau}{\sqrt{b^2+1}}, \quad \text{if} \quad k\pi < \tau < (k+1)\pi$$

$$f_3(m\pi) = q_3(0) + m\pi, \quad m = 0, 1, \ldots$$

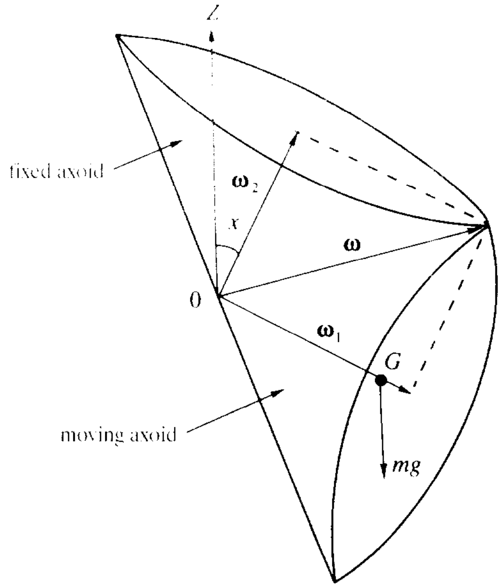

Fig. 3

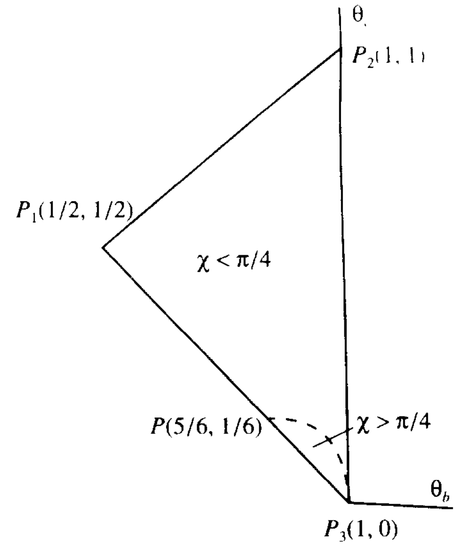

Fig. 4

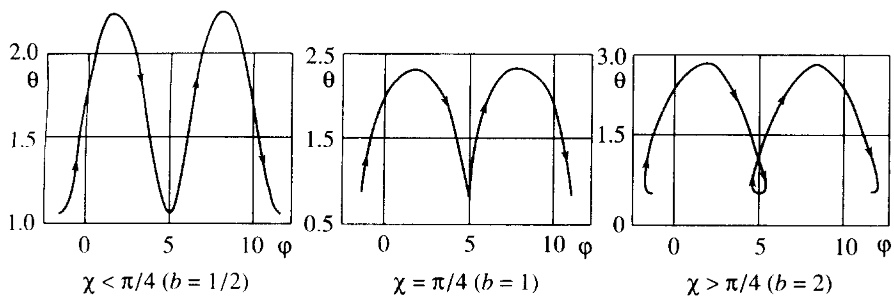

Fig. 5

In the Grioli precession, the axis of the body on which the centre of gravity is situated is an axis about which the body itself revolves, and the axis of precession is inclined to the vertical at an angle $\chi = \operatorname{arctg} b$ (Fig. 3). The axes of the moving and fixed axoids are at right angles to one another. The magnitudes of the angular velocity vectors of the body's rotation about itself ω_1 and its precession ω_2 are the same, both equalling n. The centre of gravity of the body moves in a circle whose centre lies on the axis of precession in a plane perpendicular to that axis. The motion of the body is periodic: in a time equal to the period $2\pi/n$ the body returns to its initial orientation in absolute space, the angular velocity vector then taking its initial value.

Figure 4 illustrates the dependence of the angle χ on the body's moments of inertia. The curve on which $\chi = \pi/4$ (part of a hyperbola) is shown as a dashed curve. It intersects the boundary P_1P_3 at the point $P(^5/_6, {}^1/_6)$, and the vertical straight line $\theta_b = 1$ is tangent to it at the point P_3. In the domain $\chi < \pi/4$ the angle of rotation φ of the body about itself increases monotonically ($\dot{f}_1 > 0$), but in the domain $\chi > \pi/4$ the quantity $\dot{f}_1$ may vanish and the angle φ is not monotonic. Figure 5 shows the trajectories of solution (1.6)–(1.8) corresponding to Grioli precession for different values of the inertia parameters of the body in the θ, φ plane.

2. FORMULATION OF THE STABILITY PROBLEM. DERIVATION OF THE EQUATIONS OF PERTURBED MOTION

The coordinate q_3 is cyclic. Replacing the momentum p_3 in the Hamiltonian (1.5) by its constant value from the second expression (1.8), we obtain the Hamiltonian $H(q_1, q_2, p_1, p_2; \theta_b, \theta_c)$ of the reduced system with two degrees of freedom. This system admits of a solution that is a 2π-periodic function of τ, given by Eqs (1.6) and (1.7).

We introduce perturbations Q_i and P_i, defining them, as usual, by

$$q_i = f_i(\tau) + Q_i, \quad p_i = g_i(\tau) + P_i, \quad i = 1, 2 \tag{2.1}$$

Let $\mathbf{Y}(\tau)$ be the matrix of fundamental solutions of the system of equations linearized relative to Q_i, P_i ($\mathbf{Y}(0) = \mathbf{E}$, where $\mathbf{E}$ is the 4×4 identity matrix). It is found as a rule by numerical integration. The characteristic equation of the matrix $\mathbf{Y}(2\pi)$ is reciprocal

$$\rho^4 - a_1\rho^3 + a_2\rho^2 - a_1\rho + 1 = 0 \tag{2.2}$$

where a_1 is the trace of the matrix $\mathbf{Y}(2\pi)$ and a_2 is the sum of all its 2×2 principal minors.

Since the system of equations of motion with Hamiltonian $H(q_1, q_2, p_1, p_2; \theta_b, \theta_c)$ is autonomous, the characteristic equation (2.2) has a root equal to 1. But since it is reciprocal, this root is multiple, of multiplicity at least two. Therefore, the coefficients of Eq. (2.2) satisfy the equation $a_2 = 2(a_1 - 1)$ and it may be written in the form

$$(\rho - 1)^2(\rho^2 - 2a\rho + 1) = 0, \quad a = a_1/2 - 1 \tag{2.3}$$

The periodic motion (1.6), (1.7) is not isolated. It belongs to a family of periodic motions

$$q_i = F_i(\Omega(h)\tau, h), \quad p_i = G_i(\Omega(h)\tau, h), \quad \tau = nt + \tau_0 \quad (\tau_0 = nt_0) \tag{2.4}$$

where h is the constant of the integral $H = h$ = const. In the unperturbed motion (1.6), (1.7) $h = h_0 = (\theta_c + 1)/2$, where $\Omega(h_0) = 1$, $d\Omega(h_0)/dh_0 \neq 0$.

The equations of motion linearized with respect to Q_i and P_i admit of two types of solution, obtained by differentiating the functions q_i and p_i of the family (2.4) with respect to the arbitrary constant τ_0 and h. Solutions of the first type, $\partial F_i/\partial\tau_0$, $\partial G_i/\partial\tau_0$ ($i = 1, 2$), are 2π-periodic functions of τ. (By Floquet's theory of linear differential equations with periodic coefficients, this also implies the existence of the root $\rho = 1$ of the characteristic equation (2.2).) Solutions of the second type, $\partial F_i/\partial h$, $\partial G_i/\partial h$ ($i = 1, 2$), have the structure $\nu_i(\tau) + \tau\mu_i(\tau)$ ($i = 1, 2$), where $\nu_i(\tau)$ and $\mu_i(\tau)$ are 2π-periodic functions of τ. By Floquet's theory, the presence of the term $\tau\mu_i(\tau)$ implies that the matrix $\mathbf{Y}(2\pi)$ is not diagonalizable. Consequently [16], the periodic motion (1.6), (1.7) is unstable in the first approximation in Lyapunov's sense.

It will be unstable in Lyapunov's sense in the non-linear problem also. Indeed, let

$$h = h^*, \quad 0 < |h^* - h_0| \ll 1$$

To this value of h there corresponds a periodic motion q_i^*, p_i^* of the family (2.4). We will take this motion as the perturbed motion. Since $\Omega(h^*) \neq \Omega(h_0)$, it follows that as time passes the points with coordinates $q_i(\tau)$, $p_i(\tau)$ and $q_i^*(\tau)$, $p_i^*(\tau)$ in the phase space q_1, q_2, p_1, p_2 will retreat to a finite distance from one another, however close their initial positions $q_i(0)$, $p_i(0)$ and $q_i^*(0)$, $p_i^*(0)$. This implies instability in Lyapunov's sense.

Let us consider the orbital stability of the periodic motion (1.6), (1.7), that is, look for an answer to the following question: will the trajectories of perturbed motions remain for all τ in a small neighbourhood of the trajectory of unperturbed motion if their initial points are sufficiently close together?

An algorithm was worked out in [17] to construct equations of perturbed motion in the problem of the orbital stability of periodic motions. Following the approach described in [17], we shall replace the four perturbations of the coordinates and momenta Q_i, P_i ($i = 1, 2$) defined by (2.1) by three quantities ξ_2, η_1 and η_2, characterizing the deviation of the perturbed trajectories from the trajectory of unperturbed periodic motion (1.6), (1.7). To that end, we transform variables in the system with Hamiltonian $H(q_1, q_2, p_1, p_2; \theta_b, \theta_c)$, applying a transformation that is linear in ξ_2, η_1 and η_2

$$q_1, q_2, p_1, p_2 \to \xi_1, \xi_2, \eta_1, \eta_2$$

as defined by the formulae

$$\begin{aligned} q_i &= f_i(\xi_1) + a_{i1}(\xi_1)\xi_2 + a_{i2}(\xi_1)\eta_1 + a_{i3}(\xi_1)\eta_2 \\ p_i &= g_i(\xi_1) + a_{i+2,1}(\xi_1)\xi_2 + a_{i+2,2}(\xi_1)\eta_1 + a_{i+2,3}(\xi_1)\eta_2; \quad i = 1, 2 \end{aligned} \tag{2.5}$$

where $f_i(\tau)$, $g_i(\tau)$ $(i = 1, 2)$ define the unperturbed periodic motion (1.6), (1.7). The coefficients a_{ij} are 2π-periodic functions of ξ_1. They are chosen in such a way that, in the new variables, the periodic motion we are investigating may be expressed by the equalities

$$\xi_1(\tau) = \tau + \xi_1(0), \quad \eta_1 = \xi_2 = \eta_2 = 0 \tag{2.6}$$

and the transformation of the variables (2.5) is canonical and univalent. It has been shown [17] that the coefficients $a_{ij}(\xi_1)$ must be evaluated from the formulae

$$\begin{aligned}
a_{11} &= \frac{1}{\Delta}[e_1 f_1' - e_4 f_2' - 2(e_1 c_5 - e_4 c_6 + c_1 c_5^2 - c_3^2 c_4) g_2'] \\
a_{12} &= -\frac{e_3}{\Delta}, \quad a_{13} = \frac{1}{\Delta}(c_5 f_1' - 2c_4 f_2' - e_6 g_2') \\
a_{21} &= \frac{1}{\Delta}[e_5 f_1' - e_1 f_2' + 2(e_1 c_5 - e_4 c_6 + c_1 c_5^2 - c_3^2 c_4) g_1'] \\
a_{22} &= \frac{e_2}{\Delta}, \quad a_{23} = \frac{1}{\Delta}(2c_6 f_1' - c_5 f_2' + e_6 g_1') \\
a_{31} &= \frac{1}{\Delta}(2c_1 f_2' + e_1 g_1' + e_5 g_2'), \quad a_{32} = \frac{c_3}{\Delta}, \quad a_{33} = \frac{1}{\Delta}(-f_2' + c_5 g_1' + 2c_6 g_2') \\
a_{41} &= \frac{1}{\Delta}(-2c_1 f_1' - e_4 g_1' - e_1 g_2'), \quad a_{42} = -\frac{c_2}{\Delta}, \quad a_{43} = \frac{1}{\Delta}(f_1' - 2c_4 g_1' - c_5 g_2')
\end{aligned} \tag{2.7}$$

where the prime denotes differentiation with respect to ξ_1, and we have put

$$\begin{aligned}
\Delta &= c_3 f_1' - c_2 f_2' + e_3 g_1' - e_2 g_2' \\
e_1 &= c_2 c_3 - 2c_1 c_5, \quad e_2 = c_3 c_5 - 2c_2 c_6, \quad e_3 = c_2 c_5 - 2c_3 c_4 \\
e_4 &= c_2^2 - 4c_1 c_4, \quad e_5 = c_3^2 - 4c_1 c_6, \quad e_6 = c_5^2 - 4c_4 c_6
\end{aligned} \tag{2.8}$$

The transformation of the variables (2.5) involves six arbitrary constant parameters $c_1, c_2, \dots, c_6$. They must be chosen in such a way that the quantity Δ does not vanish for $0 \leqslant \xi 1 \leqslant 2\pi$.

The Hamiltonian of the perturbed motion $\Gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ is obtained from the Hamiltonian $H(q_1, q_2, p_1, p_2; \theta_b, \theta_c)$ by replacing the variables q_1, q_2, p_1, p_2 by their expressions in terms of $\xi_1, \xi_2, \eta_1, \eta_2$ according to formulae (2.5). The function Γ can be expanded in series in powers of η_1, ξ_2 and η_2

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + \dots \tag{2.9}$$

where the constant h_0 has been dropped. and Γ_k is a form of degree k in $\eta_1|^{1/2}$, ξ_2, η_2,

$$\begin{aligned}
\Gamma_2 &= \eta_1 + \varphi_2(\xi_2, \eta_2, \xi_1), \quad \Gamma_3 = \psi_1(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_3(\xi_2, \eta_2, \xi_1) \\
\Gamma_4 &= \chi(\xi_1)\eta_1^2 + \psi_2(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_4(\xi_2, \eta_2, \xi_1)
\end{aligned} \tag{2.10}$$

Here $\chi(\xi_1)$ is a 2π-periodic function of ξ_1, and φ_m and ψ_m are forms of degree m in ξ_2 and η_2 whose coefficients are 2π-periodic functions of ξ_1.

Orbital stability of the unperturbed periodic motion implies stability of the system with Hamiltonian (2.9) with respect to perturbations of η_1, ξ_2 and η_2.

3. ORBITAL STABILITY IN THE FIRST APPROXIMATION

Let $\mathbf{X}(\xi_1)$ be the matrix of fundamental solutions ($\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the 2×2 identity matrix) of the linear system whose coefficients are 2π-periodic functions of the independent variable ξ_1

$$\frac{d\xi_2}{d\xi_1} = \frac{\partial\varphi_2}{\partial\eta_2}, \quad \frac{d\eta_2}{d\xi_1} = -\frac{\partial\varphi_2}{\partial\xi_2} \tag{3.1}$$

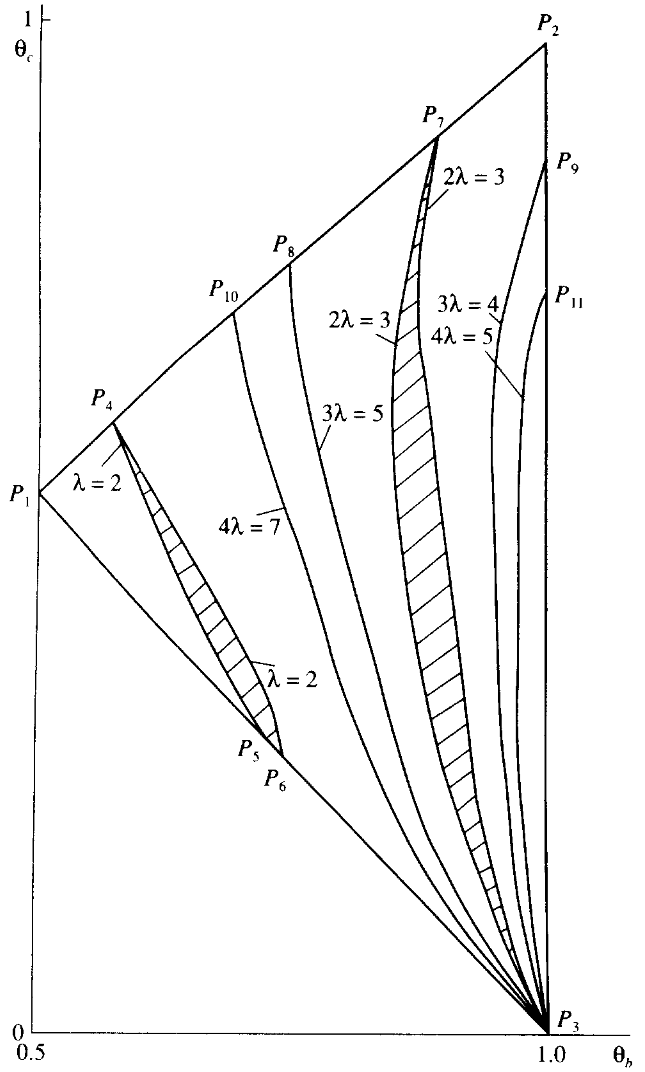

Fig. 6

where $\varphi_2(\xi_2, \eta_2, \xi_1)$ is the quadratic part (with respect to ξ_2 and η_2) of the function Γ_2 of (2.10). The characteristic equation of the matrix $\mathbf{X}(2\pi)$ is

$$\rho^2 - 2a\rho + 1 = 0, \quad a = \tfrac{1}{2}(x_{11}(2\pi) + x_{22}(2\pi)) \tag{3.2}$$

The coefficient a may also be evaluated using the second equality of (2.3).

If $|a| > 1$, Eq. (3.2) has one root, of absolute value greater than 1. There is parametric resonance, and, by Lyapunov's theorem on stability in the first approximation [16], the Grioli precession will be orbitally unstable.

The values of the parameters θ_b, θ_c for which $|a| = 1$ define the boundaries of the parametric resonance domains (PRDs).

If $|a| < 1$, the precession will be orbitally stable in the first approximation. In that case, the roots of Eq. (3.2) will be complex conjugates and their absolute value will both be equal to 1: $\rho_1 = \exp(i2\pi\lambda)$, $\rho_2 = \exp(-i2\pi\lambda)$, where λ is a real quantity defined by the equation

$$\cos 2\pi\lambda = a \tag{3.3}$$

Analytical and numerical investigations have shown that only two PRDs exist – the hatched domains in Fig. 6. They emanate from the points $P_4(\sqrt{3}/3, \sqrt{3}/3)$ and $P_7(2\sqrt{5}/5, 2\sqrt{5}/5)$ on the segment P_1P_2. In small neighbourhoods of those points, when $\theta_b = \theta_c + \varepsilon^2$, $0 < \varepsilon \ll 1$ (that is, when the rigid body differs only slightly from a dynamically symmetric body oblate along the axis of symmetry Ox'), analytical expressions have been obtained for the boundaries of the PRDs.

The left boundary of the domain emanating from the point P_4 ends at the point $P_5(0.74957, 0.25043)$ of the segment P_1P_3, and the right boundary ends at the point $P_6(0.75652, 0.24348)$. For small values of ε, the curve P_4P_5 is defined by the equation

$$\theta_b = \frac{\sqrt{3}}{3} + \frac{1}{540}(267 - 73\sqrt{3})\varepsilon^2 + \frac{1}{1\,944\,000}(278\,042\sqrt{3} - 141\,267)\varepsilon^4 + O(\varepsilon^6) =$$
$$= 0.5774 + 0.2603\varepsilon^2 + 0.1751\varepsilon^4 + O(\varepsilon^6) \tag{3.4}$$

and the curve P_4P_6 by the equation

$$\theta_b = \frac{\sqrt{3}}{3} + \frac{1}{540}(267 - 73\sqrt{3})\varepsilon^2 + \frac{1}{1\,944\,000}(453542\sqrt{3} + 1\,147983)\varepsilon^4 + O(\varepsilon^6) =$$
$$= 0.5774 + 0.2603\varepsilon^2 + 0.9946\varepsilon^4 + O(\varepsilon^6) \tag{3.5}$$

Both boundaries of the PRD emanating from the point P_7 tend, as $\theta_c \to 0$, to the corner point P_3 of the domain of admissible parameter values. Near the point P_7 the boundary curves are defined by the equations

$$\theta_b = \frac{2\sqrt{5}}{5} + \frac{1}{1000}(405 - 373\sqrt{5})\varepsilon^2 \mp \frac{1}{80}\sqrt{3050 - 1310\sqrt{5}}\varepsilon^3 + O(\varepsilon^4) =$$
$$= 0.8944 - 0.4291\varepsilon^2 \mp 0.1374\varepsilon^3 + O(\varepsilon^4) \tag{3.6}$$

where the upper and lower signs relate to the left and right boundaries, respectively.

4. NON-LINEAR ANALYSIS

If the values of the parameters θ_b and θ_c do not belong to the PRD, the first approximation is not enough for a rigorous solution of the problem of the orbital stability of the Grioli precession. The non-linear equations of perturbed motion have to be investigated.

4.1. The method of investigation

To solve the problem, formulae (2.5) and (2.7) were first used to apply a canonical transformation of variables $q_i, p_i \to \xi_i, \eta_i$ ($i = 1, 2$) so as to express the solution corresponding to Grioli precession in the form (2.6). For values of θ_b and θ_c in the domain $\chi < \pi/4$ in Fig. 4, the parameters c_k occurring in (2.5) and (2.7) were chosen as $c_1 = c_2 = c_4 = c_5 = c_6 = 0$, $c_3 = 1$. Then the transformation (2.5) becomes

$$q_1 = f_1(\xi_1), \quad q_2 = f_2(\xi_1) + \xi_2$$
$$p_1 = g_1(\xi_1) + \frac{g_2'\xi_2 + \eta_1 - f_2'\eta_2}{f_1'}, \quad p_2 = g_2(\xi_1) + \eta_2 \tag{4.1}$$

On the curve $\chi = \pi/4$ and in the domain $\chi > \pi/4$ it was assumed that $c_1 = c_2 = c_4 = c_6 = 0$, $c_3 = -c_5 = 1$, and the transformation (2.5) is written in the form

$$q_1 = f_1(\xi_1) - \eta_2, \quad q_2 = f_2(\xi_1) + \frac{f_1'\xi_2 - \eta_1 + (f_2' + g_1')\eta_2}{f_1' + g_2'}$$
$$p_1 = g_1(\xi_1) + \frac{g_2'\xi_2 + \eta_1 - (f_2' + g_1')\eta_2}{f_1' + g_2'}, \quad p_2 = g_2(\xi_1) + \eta_2 \tag{4.2}$$

In order to solve the problem of orbital stability, after transforming via formulae (4.1) or (4.2) to the variables ξ_1, ξ_2, η_1 and η_2, and obtaining a series representation (2.9) of the Hamiltonian of perturbed motion, one must obtain the normal form of the Hamiltonian of perturbed motion and then use the Arnol'd–Moser theorem [18, 19] and the stability conditions for Hamiltonian systems in the presence of resonance [20, 21]. The quadratic part of the Hamiltonian (2.9) was normalized by using the algorithms of [17, 21]; terms of higher degrees (in the present case – of degrees 3 and 4) were normalized by the Deprit–Hori method [20, 22].

The presence or absence of resonance is important in stability analysis, the most essential resonances being those of order up to and including four, that is, the cases when the number $m\lambda$, where λ is defined by Eq. (3.3) and $m = 1, 2, 3, 4$, is an integer.

Suppose the parameters θ_b and θ_c lie on the boundary of the stability domain in the linear approximation. Then $|a| = 1$. When $a = 1$ one has first-order resonance (λ is an integer), and when $a = -1$ one has second-order resonance (λ is half an integer). Let us consider the general case, in which the matrix $\mathbf{X}(\xi_1)$ of fundamental solutions of system (3.1), evaluated at $\xi_1 = 2\pi$, is not diagonalizable. Using the canonical transformation

$$\xi_1, \eta_1, \xi_2, \eta_2 \to w_1, r_1, x_2, y_2 \tag{4.3}$$

we can reduce the Hamiltonian (2.9) to the following normal form [21]

$$H = r_1 + \tfrac{1}{2}\delta y_2^2 + k_{30}x_2^3 + k_{10}x_2 r_1 + k_{40}x_2^4 + k_{20}x_2^2 r_1 + k_{00}r_1^2 + O_5 \tag{4.4}$$

where k_{ij} are constants and δ is either 1 or -1, the actual value being determined during normalization of the linear system (3.1). The symbol O_n in (4.4) (and henceforth) denotes a series beginning with terms of degree at least n in $|r_1|^{1/2}, x_2$ and y_2. The coefficients of the series O_5 in (4.4) are periodic functions of w_1. The period is 2π in first-order resonance and 4π in second-order resonance. If the coefficient k_{30} of the normal form (4.4) is not zero, or if it is zero but $\delta k_{40} < 0$, the periodic motion is orbitally unstable, but if the coefficient k_{30} in the normal form (4.4) is zero but $\delta k_{40} > 0$, the periodic motion is orbitally stable [21].

Now suppose the parameters θ_b and θ_c are in the interior of the stability domain in the first approximation. Then $|a| < 1$. If $a = -\tfrac{1}{2}$, one has third-order resonance (3λ is an integer), and if $a = 0$ one has fourth-order resonance (4λ is an integer).

We first consider the non-resonant case, in which $a \neq -\tfrac{1}{2}$ and $a \neq 0$. Then, by applying the canonical transformation (4.3), which is 2π-periodic in w_1, the Hamiltonian (2.9) may be reduced to the following normal form [17]

$$H = r_1 + \lambda r_2 + c_{20}r_1^2 + c_{11}r_1 r_2 + c_{02}r_2^2 + O_5 \tag{4.5}$$

where, from now on, $x_2 = \sqrt{2r_2}\sin w_2$, $y_2 = \sqrt{2r_2}\cos w_2$ and c_{ij} are constant coefficients. If the number defined by the equality

$$D = c_{20}\lambda^2 - c_{11}\lambda + c_{20} \tag{4.6}$$

is not zero, the periodic motion will be orbitally stable [18, 19].

For third-order resonance $3\lambda = k$, we have the following normal form of the Hamiltonian of perturbed motion [17]

$$H = r_1 + \lambda r_2 + r_2\sqrt{r_2}(\alpha_{30}\sin(3w_2 - kw_1) + \beta_{30}\cos(3w_2 - kw_1)) + O_4 \tag{4.7}$$

where α_{30} and β_{30} are constant coefficient. If

$$\alpha_{30}^2 + \beta_{30}^2 \neq 0 \tag{4.8}$$

the periodic motion will be orbitally unstable [20].

For fourth-order resonance $4\lambda = k$, the normal form of the Hamiltonian of perturbed motion will be [17]

$$H = r_1 + \lambda r_2 + c_{20}r_1^2 + c_{11}r_1 r_2 + c_{02}r_2^2 + r_2^2(\alpha_{40}\sin(4w_2 - kw_1) + \beta_{40}\cos(4w_2 - kw_1)) + O_5 \tag{4.9}$$

where α_{40} and β_{40} are constant coefficients. If

$$|D| > \sqrt{\alpha_{40}^2 + \beta_{40}^2} \tag{4.10}$$

then the periodic motion is orbitally stable [20]. But if the reverse inequality holds, the motion is orbitally unstable.

Remark. To construct normalizing transformations of the variables and compute the coefficients of the Hamiltonian of perturbed motion in normal form, one has to solve certain systems of differential equations [17, 21].

4.2. Resonances
The quantity λ is not uniquely defined by Eq. (3.3). The indeterminacy is eliminated (taking into account the continuity of the characteristic exponents) by considering the limiting case $\theta_b = \theta_c$, for which the linearized equations of perturbed motion have constant coefficients, and $\lambda = \sqrt{1 + \theta_b^{-2}}$ is the frequency of small oscillations in the neighbourhood of the trajectory of unperturbed motion.

In the curvilinear triangle $P_1P_4P_5$ in Fig. 6, $\lambda = 2 + (2\pi)^{-1}$, in the triangle $P_7P_2P_3$, $\lambda = 1 + (2\pi)^{-1}$, and in the quadrilateral $P_4P_7P_3P_6$, $\lambda = 2 - (2\pi)^{-1}$ arccos a.

On the curves P_4P_5 and P_4P_6 there is first-order resonance $\lambda = 2$, and on the boundaries of the PRD emanating from the point P_7, second-order resonance $2\lambda = 3$. In the interior of the stability domains in the first approximation there are two third-order resonance curves ($3\lambda = 5$ and $3\lambda = 4$) and two fourth-order resonance curves ($4\lambda = 7$ and $4\lambda = 5$). They begin (see Fig. 6) at the points $P_8(3/4, 3/4)$, $P_9(1, \sqrt{7}/3)$ and $P_{10}(4\sqrt{33}/33, 4\sqrt{33}/33)$, $P_{11}(1, 3/4)$, respectively. As $\theta_c \to 0$ these curves approach the point P_3.

Near the straight line $\theta_b = \theta_c$, when $\theta_b = \theta_c + \varepsilon^2$, the resonance curves $4\lambda = 7$ and $3\lambda = 5$ are defined by equations

$$\theta_b = \frac{4}{33}\sqrt{33} + \left(\frac{5221}{10\,890} - \frac{11\,074}{179\,685}\sqrt{33}\right)\varepsilon^2 + O(\varepsilon^4) =$$

$$= 0.6963 + 0.1254\varepsilon^2 + O(\varepsilon^4) \text{ (curve } 4\lambda = 7) \tag{4.11}$$

$$\theta_b = \frac{3}{4} + \frac{4747}{186\,368}\varepsilon^2 + \frac{4\,407\,170\,511\,151}{4\,741\,058\,789\,376}\varepsilon^4 + O(\varepsilon^6) =$$

$$= 0.75 + 0.0255\varepsilon^2 + 0.9296\varepsilon^4 + O(\varepsilon^6) \text{ (curve } 3\lambda = 5) \tag{4.12}$$

If $\theta_b = 1 - \varepsilon^2$, $0 < \varepsilon \ll 1$ (that is, the rigid body differs slightly from a dynamically symmetric body prolate along the symmetry axis Oz'), the equations of the resonance curves $3\lambda = 4$ and $4\lambda = 5$ will be

$$\theta_c = \frac{1}{3}\sqrt{7} - \left(\frac{9197}{32\,340}\sqrt{7} + \frac{21\,671}{10\,780}\right)\varepsilon^2 - \left(\frac{467\,489\,565\,881}{134\,220\,702\,000}\sqrt{7} + \frac{76\,509\,120\,239}{12\,782\,924\,000}\right)\varepsilon^4 + O(\varepsilon^6) =$$

$$= 0.8819 - 2.7627\varepsilon^2 - 15.2004\varepsilon^4 + O(\varepsilon^6) \text{ (curve } 3\lambda = 4) \tag{4.13}$$

$$\theta_c = \frac{3}{4} - \frac{20401}{4536}\varepsilon^2 - \frac{11\,809\,141\,307}{216\,040\,608}\varepsilon^4 + O(\varepsilon^6) =$$

$$= 0.75 - 4.4976\varepsilon^2 - 54.6617\varepsilon^4 + O(\varepsilon^6) \text{ (curve } 4\lambda = 5) \tag{4.14}$$

4.3. Results
We will now describe the results of analytical and numerical investigation of the orbital stability of the Grioli precession for values of the parameters θ_b and θ_c not in the PRD. These results are illustrated graphically in Fig. 7.

We will first consider the two aforementioned cases of a body that is nearly dynamically symmetric. In these cases the orbital stability problem for the Grioli precession can be investigated by analytical means. This has been done using the MAPLE VII system.

1. *The case* $\theta_b = \theta_c + \varepsilon^2$ $(0 < \varepsilon \ll 1)$. In the limit when $\varepsilon = 0$, the body is dynamically symmetric and its mass geometry ($A > B = C$) corresponds to the segment P_1P_2 of the boundary of the domain of admissible values of the parameters θ_b and θ_c. For points (θ_b, θ_c) not on one of the resonance curves

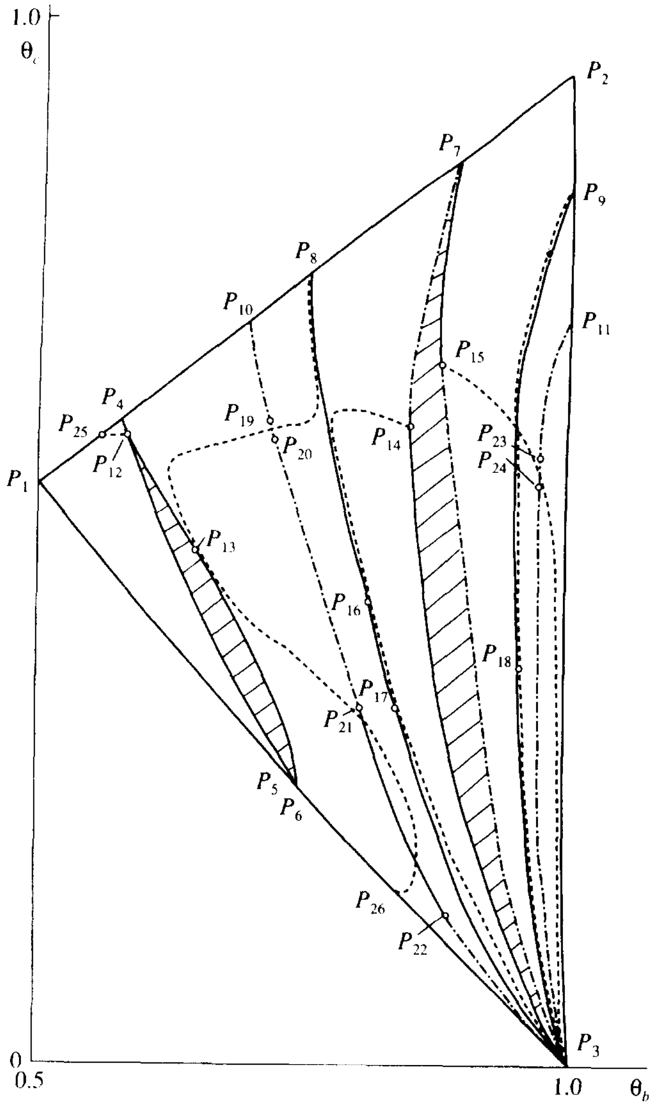

Fig. 7

emanating from the points P_4, P_{10}, P_8, P_7 of the segment P_1P_2 the Hamiltonian of perturbed motion has the normal form (4.5). The following expressions may be obtained for the coefficients of the normal form

$$\lambda = \frac{\sqrt{\theta_b^2+1}}{\theta_b} + O(\varepsilon^2), \quad c_{20} = \frac{\theta_b^2 - \theta_b + 1}{2(\theta_b^2+1)} + O(\varepsilon^2)$$
$$c_{11} = \frac{1-2\theta_b^2}{\theta_b(\theta_b^2+1)\sqrt{\theta_b^2+1}} + O(\varepsilon^2), \quad c_{02} = \frac{2\theta_b^4 - 11\theta_b^2 + 2}{4\theta_b(\theta_b^2+1)^2} + O(\varepsilon^2) \tag{4.15}$$

The quantity (4.6) will be

$$D = \frac{2\theta_b^6 + 14\theta_b^4 - 15\theta_b^3 + 10\theta_b^2 - 2}{4\theta_b^2(\theta_b^2+1)^2} + O(\varepsilon^2) \tag{4.16}$$

When $\varepsilon = 0$ the number D vanishes only once in the interval P_1P_2 – at the point $P_{25}(0.56776, 0.56776)$.

The interval P_1P_2 is divided by the points P_{25}, P_4, P_{10}, P_8, P_7 into six intervals. Suppose the values θ_c^* and θ_c^{**} ($\theta_c^* < \theta_c^{**}$) correspond to the boundary points of one of these intervals. Then for small ε, if

$$\theta_c^* + f^*(\varepsilon) < \theta_c < \theta_c^{**} - f^{**}(\varepsilon) \tag{4.17}$$

where f^* and f^{**} are certain continuous positive functions of ε that vanish when ε = 0, there will be no resonances of order up to and including four, and the number D will not vanish. According to the discussion in Section 4.1, this implies that, if ε is sufficiently small and inequality (4.17) holds, the Grioli precession is orbitally stable. In other words, for small values of ε, each of the aforementioned six subintervals of the interval P_1P_2 is adjacent to an orbital stability domain.

We will now consider the resonant cases.

First-order resonance $\lambda = 2$. On the curves P_4P_5 and P_4P_6 defined for small values of ε Eqs (3.4) and (3.5), respectively, the normal form of the Hamiltonian of perturbed motion may be written as (4.4). It can be shown that on the curve P_4P_5 the normal form (4.4) will have $\delta = 1$, $k_{30} = 0$,

$$k_{40} = \left(\frac{179511837}{671088640} - \frac{1849015981}{5033164800}\sqrt{3}\right)\varepsilon^{12} + O(\varepsilon^{13}) = -0.3688\varepsilon^{12} + O(\varepsilon^{13})$$

while on the curve P_4P_6 we have $\delta = -1$, and

$$k_{30} = \sqrt[4]{12}\left(\frac{3464297}{15728640} - \frac{35449069}{47185920}\sqrt{3}\right)\varepsilon^{10} + O(\varepsilon^{12}) = -2.0119\varepsilon^{10} + O(\varepsilon^{12})$$

Therefore, on both the left and right boundaries of the PRD emanating from the point P_4, the Grioli precession is orbitally unstable for sufficiently small ε.

Second-order resonance $2\lambda = 3$. On the boundaries (3.6) of the PRD emanating from the point P_7, the normal form of the Hamiltonian of perturbed motion, as in the case of first-order resonance, will have the form (4.4); on both the left and right boundaries

$$k_{30} = 0, \quad k_{40} = \frac{25}{3981312}(167065 - 74123\sqrt{5})\varepsilon^6 + O(\varepsilon^7) = 0.0083\varepsilon^6 + O(\varepsilon^7)$$

but $\delta = 1$ on the left boundary and $\delta = -1$ on the right.

Hence, by the discussion in Section 4.1, it follows that for sufficiently small ε the Grioli precession will be orbitally stable on the left boundary of the PRD emanating from P_7 and unstable on the right.

Third-order resonance $3\lambda = 5$. The resonance curve is defined by Eq (4.12). On that curve the normal form of the Hamiltonian is given by (4.7), with

$$\alpha_{30} = -\frac{1270053}{10976000}\sqrt{10}\varepsilon^5 + O(\varepsilon^6) = -0.3659\varepsilon^5 + O(\varepsilon^6), \quad \beta_{30} = 0$$

For sufficiently small ε, inequality (4.8) holds, and this is therefore a case of orbital instability.

Fourth-order resonance $4\lambda = 7$. On the corresponding curve (4.11), the normal form of the Hamiltonian (4.9) will have $\alpha_{40} = 0$, $\beta_{40} = O(\varepsilon^2)$, and the quantity D of (4.6) is expressed as

$$D = \frac{2335}{1568} - \frac{495}{2401}\sqrt{33} + O(\varepsilon^2) = 0.3048 + O(\varepsilon^2)$$

Since inequality (4.10) holds for sufficiently small ε, it follows that the Grioli precession is orbitally stable.

2. *The case* $\theta_b = 1 - \varepsilon^2$ ($0 < \varepsilon \ll 1$). When ε = 0 the body is dynamically symmetric, and its mass geometry ($A = B > C$) corresponds in Fig. 7 to the vertical segment P_2P_3 of the boundary of the domain of admissible values of the parameters θ_b and θ_c. If the parameters θ_b and θ_c do not belong to the curves (4.13) and (4.14) emanating from the points P_9 and P_{11} of the segment P_2P_3, then for sufficiently small ε there is no resonance of order up to and including four. The normalized Hamiltonian has the form (4.5), and its coefficients are evaluated using the formulae

$$\lambda = \sqrt{\theta_c^2+1}+O(\varepsilon^2), \quad c_{20} = \frac{\theta_c^2-\theta_c+1}{2\theta_c(\theta_c^2+1)}+O(\varepsilon^2)$$
$$c_{11} = \frac{\theta_c(\theta_c^2-2)}{(\theta_c^2+1)\sqrt{\theta_c^2+1}}+O(\varepsilon^2), \quad c_{02} = \frac{2\theta_c^4-11\theta_c^2+2}{4(\theta_c^2+1)^2}+O(\varepsilon^2) \tag{4.18}$$

For D we obtain the expression

$$D = -\frac{2\theta_c^6-10\theta_c^4+15\theta_c^3-14\theta_c^2-2}{4\theta_c(\theta_c^2+1)^2}+O(\varepsilon^2)$$

When $\varepsilon = 0$ the number D does not vanish for any values of θ_c in the interval P_2P_3.

The points P_9 and P_{11} divide this interval into three subintervals (P_2P_9), (P_9P_{11}) and $(P_{11}P_3)$. Let θ_c' and θ_c'' $(\theta_c' < \theta_c'')$ denote the boundary points of any of these intervals. Then for small values of ε, the domain defined by the inequalities

$$\theta_c'+f'(\varepsilon)<\theta_c<\theta_c''-f''(\varepsilon) \tag{4.19}$$

where f' and f'' are certain continuous positive functions of ε that vanish at $\varepsilon = 0$, contains no resonances of order up to and including four, and $D \neq 0$. By the discussion in Section 4.1, the Grioli precession will be orbitally stable in the domain (4.19). This means that each of the three intervals (P_2P_9), (P_9P_{11}) and $(P_{11}P_{13})$ is adjacent to an orbital stability domain.

We will now consider the resonant cases, when the parameters θ_b and θ_c belong to the curves (4.13) and (4.14).

Third-order resonance $3\lambda = 4$. On the corresponding resonance curve (4.13) the normal form of the Hamiltonian is (4.7), with $\beta_{30} = 0$ and

$$\alpha_{30} = \frac{1}{1024000}\sqrt{6}(3249715-1128212\sqrt{7})\varepsilon^4+O(\varepsilon^5) = 0.6333\varepsilon^4+O(\varepsilon^5)$$

Since inequality (4.8) holds for sufficiently small ε, this implies orbital instability.

Fourth-order resonance $4\lambda = 5$. On the curve (4.14) the normalized Hamiltonian of perturbed motion is (4.9), with $\alpha_{40} = 0$, $\beta_{40} = O(\varepsilon^2)$, and

$$D = \frac{2603}{3000}+O(\varepsilon^2) = 0.8677+O(\varepsilon^2)$$

For small ε values, inequality (4.10) holds, and the Grioli precession is orbitally stable.

3. *Arbitrary values of the parameters*. For arbitrary values of the parameters θ_b and θ_c in the domain of admissible values, the coefficients of the normal form of the Hamiltonian needed to investigate stability were found by numerically. In accordance with the algorithms presented in [17, 21], this required the integration of certain systems of ordinary differential equations with previously known initial conditions. The computations were carried out for values of θ_c not less than 0.01. The results will now be described (see also Fig. 7).

On the left boundary P_4P_5 of the PRD emanating from the point P_4, Grioli precession is orbitally unstable everywhere except at the point $P_{12}(0.578, 057175)$, where the question of stability remains open. On the right boundary P_4P_6 one also has instability, everywhere except at the point $P_{13}(0.65635, 0.444957)$, where precession is orbitally stable.

The left boundary of the PRD emanating from the point P_7 is divided by the point $P_{14}(0.853, 0.604)$ into two parts. On the part P_7P_{14} one has orbital stability, at the point P_{14} the question of stability remains open, and at all other points investigated precession is unstable. The right boundary is also divided by the point $P_{15}(0.87876, 0.678)$ into two parts. Adjoining the point P_7 is a segment P_7P_{15} of orbital instability, at the point P_{15} the question of stability remains open, and at all other points investigated one has orbital stability.

The parts of the resonance curves on which Grioli precession is orbitally unstable are shown in Fig. 7 by solid lines, and the parts on which it is orbitally stable are shown by dash–dot lines.

On the third-order resonance curves there is always orbital instability, except at the points $P_{16}(0.809339, 0.449)$ and $P_{17}(0.831305, 0.336)$ on the curve $3\lambda = 5$ and the point $P_{18}(0.954319, 0.389)$ on the curve $3\lambda = 4$, where there is orbital stability.

On the fourth-order resonance curve $4\lambda = 7$, unstable segments $P_{19}P_{20}$ and $P_{21}P_{22}$ were observed. At the boundary points of these segments $P_{19}(0.7221, 0.5905)$, $P_{20}(0.7224, 0.5895)$, $P_{21}(0.809, 0.332)$, $P_{22}(0.892, 0.138)$ the question of stability remains open. At other points investigated the precession is orbitally stable.

On the curve $4\lambda = 5$ there is an unstable segment $P_{23}P_{24}$. At its boundary points $P_{23}(0.97688, 0.5746)$ and $P_{24}(0.9767, 0.5717)$ the question of stability remains open. At other points of this curve that were investigated, the precession is orbitally stable.

For values of the parameters θ_b and θ_c outside the PRD and not on the resonance curves of order up to and including four, the Grioli precession is orbitally stable everywhere except possibly on the curve $D = 0$, where the condition of the Arnol'd–Moser theorem breaks down. This curve consists of five parts, shown in Fig. 7 as dashed curves: a part passing through the points P_{25} and P_{12}, the part connecting the points P_8 and $P_{26}(0.83902, 0.16098)$, the parts $P_{14}P_3$ and $P_{15}P_9$, and the loop-shaped part between the curve $3\lambda = 4$ and the vertical $\theta_b = 1$.

Thus, the problem of the orbital stability of the Grioli precession has been solved for almost all admissible values of the parameters in the domain $\theta_b \geqslant 0.01$; for the remaining six uninvestigated points $P_{k+18}(k = 1, 2, \ldots, 6)$ on the fourth-order resonance curves, and for the curve $D = 0$, an analysis of the stability requires a consideration of terms of order greater than four in the series expansion of the Hamiltonian of perturbed motion.

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REFERENCES

1. GRIOLI, G., Esistenza e determinazione delle precessioni regolari dinamicamente possibili per un solido pesante asimmetrico. *Ann. Mat. Pura ed Appl., Ser. 4*, 1947, **26**, 3–4, 271–281.
2. ROUTH, E. J., *Dynamics of a System of Rigid Bodies*, Part 2. Dover, New York, 1955.
3. GRAMMEL, R., *Der Kreisel, seine Theorie und seine Anwendungen*, Vol. 1. Springer, Berlin, 1950.
4. GULYAYEV, M. P., A new particular solution of the equations of motion of a heavy rigid body having one fixed point. *Vestnik Mosk. Gos. Univ., Ser. Fiz-Mat. i Yestestv. Nauk*, 1955, 3, 15–21.
5. GULYAYEV, M. P., Dynamically possible regular precessions of a rigid body having one fixed point. *Trudy Sektora Mat. i. Mekh. Akad. Nauk KazSSR*, 1958, **1**, 202–208.
6. GULYAYEV, M. P., Circular sections of reciprocal ellipsoids of inertia. *Trudy Sektora Mat. i Mekh. Akad. Nauk KazSSR*, 1958, **1**, 175–193.
7. GULYAYEV, M. P., Regular precession of an asymmetric gyroscope the Grioli case). In *Theoretical Mechanics. Collection of Scientific and Methodological Papers*, No. 5. Nauka, Moscow, 1975, pp. 130–137.
8. GORR, G. V., Methods of investigating the motions of a rigid body and their applications in the classification of motions. In *Mechanics of Solids*, No. 14. Naukova Dumka, Kiev, 1982, pp. 54–74.
9. GALIULLIN, I. A., Regular precessions of a rigid body with one fixed point. *Izv. Akad. Nauk SSSR. MTT*, 1987, 5, 6–18.
10. GALIULLIN, I. A., History of the discovery and investigation of regular precessions of a rigid body. In *Studies in the History of Physics and Mechanics*, 1993–1994. Nauka, Moscow, 1997, pp. 191–218.
11. GRIOLI, G., Questioni di stabilita reguardanti le precessioni regolari del solido pesante asimmetrico. *Ann. Scuola Norm. Super. Pisa, Ser. 3*, 1949, **1**, 43–71.
12. BRYUM, A. Z., Investigation of the regular precession of a heavy rigid body with a fixed point by Lyapunov's first method. In *Mechanics of Solids*, No. 19. Naukova Dumka, Kiev, 1987, pp. 68–72.
13. MOZALEVSKAYA, G. V., KHARLAMOV, A. P. and KHARLAMOVA, Ye. I., Drift of G. Grioli's gyroscope. In *Mechanics of Solids*, No. 24. Naukova Dumka, Kiev, 1992, pp. 15–25.
14. TKHAI, V. N., The stability of regular Grioli precessions. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 848–857.
15. MARKEYEV, A. P., The stability of the regular precessions of an asymmetric gyroscope (the Grioli case). *Dokl. Ross. Akad. Nauk*, 2002, **387**, 3, 338–342.
16. MALKIN, I. G., *Theory of Stability of Motion*. Nauka, Moscow, 1966.
17. MARKEYEV, A. P., An algorithm for normalizing of a Hamiltonian system in the problem of the orbital stability of periodic motions. *Prikl. Mat. Mekh.*, 2002, **66**, 6, 929–938.
18. ARNOL'D, V. I., KOZLOV, V. V. and NEISHTADT, A. I., *Mathematical Aspects of Classical and Celestial Mechanics*. Editorial URSS, Moscow, 2002.
19. MOSER, J. K., *Lectures on Hamiltonian Systems*. American Mathematical Society, Providence, RI, 1968.
20. MARKEYEV, A. P., *Libration Points in Celestial Mechanics and Cosmodynamics*. Nauka, Moscow, 1978.
21. MARKEYEV, A. P., Investigation of the stability of the periodic motions of an autonomous Hamiltonian system in a critical case. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 833–847.
22. GIACAGLIA, G. E. O., *Perturbation Methods in Non-linear Systems*. Springer, New York, 1972.

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